J Math Chem (2014) 52:178–187
DOI 10.1007/s10910-013-0253-5

ORIGINAL PAPER

A new approach to maturity of molecules by rationality of finite groups

Reza Zahed · Hesam Sharifi

Received: 22 March 2013 / Accepted: 17 August 2013 / Published online: 29 August 2013

Abstract These two concepts, maturity in chemistry and rationality in group theory were discovered by a chemist, Fujita. In the present study, we introduce a new approach to maturity and immaturity of simple groups, using the deep theorem (Feit and Seitz in Ill J Math 33:101–131, 1988). Additionally, we prove that 1,3,5-trimethyl-2,4,6-trinitrobenzene are always unmatured and tetra platinum(II) with point group D_{2n}, dihedral group of order $2n$, is unmatured if $n \neq 1, 2, 3, 4, 6$. Also, we compute integer-valued characters of the simple sporadic group Ly.

Keywords Rational group · Integer-valued characters · Matured groups · Dominant classes · Markaracter · Lyons group

1 Introduction

In recent years, the problems of group theory have attracted the wide attention of researchers in mathematics, physics and chemistry. Many problems of the computational group theory have been solved, such as the classification of simple groups, the symmetry of molecules, etc. It is not only on the property of finite group, but also its wide-ranging connection with many applied sciences, such as nanoscience, chemical physics and quantum chemistry, biomedical are areas of active research in group theory, for instant see [4,6–8,10,11].

R. Zahed
Department of Emergency Medicine, Imam Khomeini Hospital Complex, Faculty of Medicine, Tehran University of Medical Sciences, Tehran, Iran
e-mail: rzahed@gmail.com

H. Sharifi (✉)
Department of Mathematics, Faculty of Science, Shahed University, Tehran, Iran
e-mail: hsharifi@shahed.ac.ir

The matured and unmatured groups were introduced by famous chemist Fujita. He used character theory of finite groups in calculating mark table and $\mathbb{Q}$-conjugacy character. They are applied to combinatorics enumeration of isomers of molecules.

By the Theorem 2.2 in this paper, Lyons group of order 51765179004000000 is an unmatured group. The motivation for this study is outlined in [6,11] and [16], and the reader is encouraged to consult the papers [1,12,13] and [14] for background material as well as basic computational techniques.

We prepared the article as follows: In Sect. 2, we introduced some necessary concepts, such as the maturity, $\mathbb{Q}$-group and $\mathbb{Q}$-conjugacy character of a finite group. In Sect. 3, we provided Examples 3.4 and 3.5 of unmatured groups and computed the dominant classes and $\mathbb{Q}$-conjugacy characters for the Lyons group.

2 Preliminaries

Throughout this paper we adopt the same notations as in ([6,11]). We will use the $\mathbb{ATLAS}$ of finite groups notations [1] for conjugacy classes. Thus, nx, n is an integer and $x = a, b, c, \ldots$ denote conjugacy classes of G of elements of order n.

Before stating discussion, we will mention some well-known results about $\mathbb{Q}$-conjugation, where $\mathbb{Q}$ denotes the field of rational numbers. An alternative characterization of $\mathbb{Q}$-conjugation is the following concepts which can be found in [3–5,7,9].

A *dominant class* is defined as a disjoint union of conjugacy classes that correspond to the same cyclic subgroup, which is selected as a representative of conjugate cyclic subgroups. Let G be a finite group and $h_1, h_2 \in G$. We say h_1 and h_2 are $\mathbb{Q}$-conjugate if $t \in G$ exists such that $t^{-1} < h_1 > t =< h_2 >$ which is an equivalence relation on group G and generates equivalence classes that are called dominant classes. The group G is partitioned in to dominant classes as follows: $G = K_1 + K_2 + \ldots + K_s$ in which K_i corresponding to the cyclic (dominant) subgroup G_i selected from a non-redundant set of cyclic subgroups of G denoted by $SCSG$.

Suppose C be a $m \times m$ matrix of the character table for an arbitrary finite group G. Then, C is transformed into a more concise form called the $\mathbb{Q}$-conjugacy character table denoted by $C_G^{\mathbb{Q}}$ containing integer-valued characters. According to theorem 4 in [6], the dimension of a $\mathbb{Q}$-conjugacy character table, $C_G^{\mathbb{Q}}$ is equal to its corresponding markaracter table, i.e., $C_G^{\mathbb{Q}}$ is an $n \times n$-matrix where n is the number of dominant classes or equivalently the number of $SCSG$. If $m = n$, then $C = C^{\mathbb{Q}}$ i.e. G is a *maturated* group. Otherwise, $n < m$ (is called *unmaturated* group) for each $G_i \in SCGG$ (the corresponding dominant class K_i) set $t_i = m(G_i)/\varphi(|G_i|)$ where $m(G_i) = |N_G(G_i)|/|C_G(G_i)|$ (called the maturity discriminant), where the symbols $N_G(G_i)$ denotes the normalizer of G_i in G and $C_G(G_i)$ is centralizer G_i in G, also, φ is the Euler function. If $t_i = 1$ then, K_i is exactly a conjugacy class so there is no reduction in row and column of C but if $t_i > 1$ then K_i is a union of t_i-conjugacy classes of G (i.e. reduction in column) therefore the sum of t_i rows of irreducible characters via the same degree in C (reduction in rows) gives us a reducible character which is called the $\mathbb{Q}$-conjugacy character.

Now, we need to recall some concepts of rational group theory. Let G be a finite group and χ be a complex character of G. If for every $x \in G$ we have $\chi(x) \in \mathbb{Q}$, by definition, χ is called rational character. A finite group G is called a rational group or a $\mathbb{Q}$-group, if all irreducible complex characters of G are rational. For example, the symmetric group S_n and the Weyl groups of the classical complex Lie algebras are rational groups (for more details see [1]). A comprehensive description of rational groups can be found in [15].

Theorem 2.1 ([15]) *A group G is a* $\mathbb{Q}$*-group if and only if for every* $x \in G$ *of order* n *the elements* x *and* x^m *are conjugacy in G, whenever* $(m, n) = 1$.

Equivalently, for each $x \in G$ *we must have* $\frac{N_G(<x>)}{C_G(<x>)} \simeq Aut(<x>)$.

The following deep Theorem due to Fiet and Siet [2].

Theorem 2.2 *Let G be a noncyclic simple group. Then G is a* $\mathbb{Q}$*-group if and only if* $G \simeq Sp_6(2)$ *or* $O_8^+(2)'$.

3 Results and discussions

By Definition $\mathbb{Q}$-conjugacy class and Theorems 2.1 and 2.2, every $\mathbb{Q}$-group is matured. Thus we have the first result:

Result 3.1 *Let G be a finite group, then G is* $\mathbb{Q}$*-group if and only if it is matured.*

In structure of finite $\mathbb{Q}$-groups, we have the following important results, in fact this is our new approach.

Result 3.2 *Let G be a non-trivial* $\mathbb{Q}$*-group. Then:*

(1) If p is a prime divisor of $|G|$*, then* $p - 1||G|$.
(2) A quotient group G is a $\mathbb{Q}$*-group.*
(3) The direct product (denotes $\times$*) and wreath product (denotes* wr*) of a finite number of* $\mathbb{Q}$*-groups is a* $\mathbb{Q}$*-group, and vice versa.*

Proof For its proof see [15].

□

Result 3.3 *Matured groups are always of even order.*

Proof By the Results 3.1 and 3.2, part (1), it is obvious. □

Example 3.4 Point group tetra platinum(II) is D_{2n} dihedral group of order $2n$. By using the character table D_{2n} is $\mathbb{Q}$-group iff $n = 1, 2, 3, 4, 6$. Therefore, tetra platinum(II) is unmatured if and only if $n \neq 1, 2, 3, 4, 6$.

Example 3.5 The full non-rigid (f-NRG) group of 1,3,5-trimethyl-2,4,6-trinitrobenzene is isomorphic to the group $(\mathbb{Z}_2 \times \mathbb{Z}_3) wr S_3$ of order 1296, where $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are cyclic groups of order 2 and 3, respectively and S_3 is the symmetric group of order 6 on 3 letters. By the Result 3.2 the group is unmatured, because $\mathbb{Z}_3$ is unmatured.

According to Theorem 2.2, the Lyons group *Ly* is an unmatured group. Now we are equipped to compute all the dominant classes and $\mathbb{Q}$-conjugacy characters for the above group, using a GAP program [12].[1]

Theorem 3.6 *The Lyons group Ly has thirty-nine dominant classes, among which ten dominant classes are unmatured. Moreover, the unmaturated dominant classes of Ly have orders 11, 21, 22, 24, 31, 33, 37, 40, 42 and 67 with the corresponding maturities 2, 2, 2, 2, 5, 2, 2, 2, 2 and 3, respectively.*

Proof The dimension of a $\mathbb{Q}$-conjugacy character table, $C_{Ly}^{\mathbb{Q}}$ is equal to its corresponding markaracter table for Ly. To find the number of dominant classes, at first, we calculate the table of marks for *Ly* [13,14] via GAP system, see GAP programs in [12] for more details. Hence, the markaracter table for *Ly* includes ten non-conjugate cyclic subgroups(i.e., $G_i \in SCS_{Ly}$) of orders 11, 21, 22, 24, 31, 33, 37, 40, 42 and 67.

Therefore, by using the above table, the character table of Ly and definition of dominant class, since $|SCS_{Ly}| = 10$, the dominant classes of *Ly* are $A_{11} = 11a \cup 11b$, $B_n = na \cup nb$ for $n = 21, 22, 33, 37, 40, 42$, $C_{24} = 24b \cup 24c$ for and $D_{31} = 31a \cup 31b \cup 31c \cup 31d \cup 31e$ and $E_{67} = 67a \cup 67b \cup 67c$ with maturity (i.e., $t = \varphi(n)/m(H)$) 2, 2, 2, 2, 5, 2, 2, 2, 2 and 3, respectively. □

The Lyons group *Ly* has ten unmatured $\mathbb{Q}$-conjugacy characters. Furthermore, *Ly* has ten unmatured $\mathbb{Q}$-conjugacy characters $\chi_2, \chi_4, \chi_5, \chi_{18}, \chi_{20}, \chi_{21}, \chi_{22}, \chi_{23}, \chi_{30}$ and χ_{34} which are the sum of some irreducible characters. Indeed, if $Irr(Ly) = \{\varphi_1, \ldots, \varphi_{53}\}$ is the set of all irreducible characters of Ly, then integer-valued characters are the following $Irr_{\mathbb{Q}}(Ly) = \{\chi_1, \ldots, \chi_{39}\}$, such that:

$$\begin{aligned}
&\chi_2 = \varphi_2 + \varphi_3,\ \chi_4 = \varphi_5 + \varphi_6,\ \chi_5 = \varphi_7 + \varphi_8 \\
&\chi_{18} = \varphi_{21} + \varphi_{22},\ \chi_{20} = \varphi_{24} + \varphi_{25},\ \chi_{21} = \varphi_{26} + \varphi_{27} + \varphi_{28} \\
&\chi_{22} = \varphi_{29} + \varphi_{30},\ \chi_{23} = \varphi_{31} + \varphi_{32},\ \chi_{34} = \varphi_{47} + \varphi_{48}\ \chi_{30} = \varphi_{39} + \varphi_{40} \\
&\quad + \varphi_{41} + \varphi_{42} + \varphi_{43}
\end{aligned}$$

where all the rest of characters do not change. Therefore, there are ten column-reductions (similarly ten row-reductions) in the character table of Ly [6,11].

We provide all $\mathbb{Q}$-conjugacy characters of *Ly* in Tables 1 and 2.

[1] which is available freely from: http://www.gap-system.org.

Table 1 The integer-valued character table of Lyons group Ly where $A_{11} = 11a \cup 11b$ is an unmatured dominat class

$C_{Ly}^{\mathbb{Q}}$	1a	2a	3a	3b	4a	5a	5b	6a	6b	6c	7a	8a
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	4,960	−32	208	−8	0	−40	10	16	−8	4	4	0
χ_3	45,694	110	253	10	26	69	−6	29	2	2	−2	4
χ_4	96,348	252	714	12	−28	98	−2	42	12	0	0	−8
χ_5	240,128	0	−1,792	−64	0	128	28	0	0	0	0	0
χ_6	381,766	−154	−770	67	14	−109	−9	14	11	−1	0	−6
χ_7	1,152,735	−417	2,751	51	−1	235	10	63	3	−3	3	−1
χ_8	1,534,500	660	1,980	117	16	125	0	−36	−3	3	2	6
χ_9	3,028,266	1,242	5,103	0	6	141	16	63	0	0	3	4
χ_{10}	3,073,960	−1,112	5,356	10	8	210	10	−20	10	4	1	0
χ_{11}	4,226,695	759	−209	115	35	−180	−5	111	3	3	4	5
χ_{12}	4,997,664	672	8,064	−36	0	164	14	0	12	6	0	0
χ_{13}	5,379,430	−826	7,294	31	14	55	5	14	−1	−7	0	6
χ_{14}	10,758,860	−308	−9,604	−46	28	110	10	28	−14	−2	0	0
χ_{15}	11,834,746	−1,078	−1,001	106	14	371	−4	119	2	2	0	−4
χ_{16}	16,906,780	44	6,292	−107	16	−95	5	20	5	−1	2	−6
χ_{17}	18,395,586	594	−16,038	0	6	−39	11	90	0	0	−1	−4
χ_{18}	36,791,172	1,188	16,038	0	12	−78	22	−90	0	0	−2	−8
χ_{19}	19,212,250	330	−5,270	49	22	−250	0	−6	9	−3	1	0
χ_{20}	42,625,000	−2,200	5,500	100	40	0	0	−100	20	−4	−2	0

Table 1 continued

$C_{Ly}^{\mathbb{Q}}$	1a	2a	3a	3b	4a	5a	5b	6a	6b	6c	7a	8a
χ_{21}	67,828,992	0	−29,568	240	0	−1,008	42	0	0	0	0	0
χ_{22}	54,505,440	3,168	4,752	216	0	440	−10	−48	−24	0	−4	0
χ_{23}	54,505,440	3,168	−9,504	−108	0	440	−10	96	12	0	−4	0
χ_{24}	28,787,220	−924	−2,772	63	56	345	−5	−84	−9	3	0	−6
χ_{25}	29,586,865	385	15,169	−32	21	−635	−10	49	8	−2	0	−1
χ_{26}	30,739,600	−1,200	6,040	−8	0	−400	0	120	0	6	−4	0
χ_{27}	33,813,560	2,552	11,396	2	8	−190	10	−28	2	−4	−3	0
χ_{28}	36,887,520	−1,440	16,752	12	0	20	20	48	−12	0	−2	0
χ_{29}	38,734,375	−825	−10,625	40	15	0	0	15	0	−6	1	−5
χ_{30}	215,550,720	0	0	0	0	720	−30	0	0	0	0	0
χ_{31}	44,159,500	−1,540	1,540	−161	56	125	0	−28	−1	−1	0	6
χ_{32}	45,648,306	−990	−20,790	−54	−34	181	6	42	−6	0	−3	4
χ_{33}	45,648,306	−990	10,395	108	−34	181	6	−21	12	0	−3	4
χ_{34}	91,388,000	1,760	7,040	−88	0	500	0	128	8	−4	4	0
χ_{35}	52,994,655	−945	5,103	0	−21	−345	−20	63	0	0	0	1
χ_{36}	53,765,625	−375	8,625	−15	−35	0	0	−15	−15	3	4	−5
χ_{37}	56,022,921	297	−5,103	0	−15	−204	−4	−63	0	0	3	5
χ_{38}	64,906,250	−550	−1,750	5	−50	0	0	−70	5	−1	3	0
χ_{39}	71,008,476	924	−10,164	−66	28	−274	1	−84	6	6	0	0

Table 1 continued

$C_{Ly}^{\mathbb{Q}}$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
8b	1	0	0	0	0	2	−1	2	0	0	1	0	−2	0	0	−2	0	0	−4	0
9a	1	−2	1	0	−4	1	0	0	0	1	−2	0	1	−1	1	1	0	0	−2	−2
10a	1	8	5	2	0	11	3	5	−3	18	4	12	−1	22	−13	9	9	18	−10	0
10b	1	−2	0	2	0	1	−2	0	2	−2	−1	2	−1	2	2	−1	−1	−2	0	0
A11	1	−1	0	−1	−2	0	1	0	−1	−1	0	1	1	2	0	0	0	0	2	0
12a	1	0	5	2	0	2	−1	4	3	−4	−1	0	2	4	−1	4	−6	6	−2	4
12b	1	0	2	−4	0	−1	−1	1	0	2	−1	0	−1	−2	2	1	0	0	1	4
14a	1	−4	−2	0	0	0	3	2	3	1	−4	0	0	0	0	2	−1	−2	1	−2
15a	1	8	3	14	8	5	1	5	3	6	−9	14	19	−4	−1	−8	12	−12	5	0
15b	1	2	0	2	−4	2	1	2	0	0	0	−1	1	−1	−4	−2	0	0	−1	0
15c	1	−2	3	4	−2	0	1	0	−2	1	1	−1	−1	1	−1	2	2	−2	0	0
18a	1	−2	−1	0	0	−1	0	0	0	1	0	0	−1	1	−1	−1	0	0	0	2

Table 1 continued

$C_{Ly}^{\mathbb{Q}}$	8b	9a	10a	10b	A11	12a	12b	14a	15a	15b	15c	18a
χ_{21}	0	6	0	0	0	0	0	0	−18	0	−3	0
χ_{22}	0	0	8	−2	0	0	0	4	2	−4	2	0
χ_{23}	0	0	8	−2	0	0	0	4	−4	2	−4	0
χ_{24}	−2	0	1	1	0	−4	−1	0	3	3	−2	0
χ_{25}	3	1	5	0	0	−3	0	0	19	−2	−1	1
χ_{26}	0	−2	0	0	1	0	0	4	−10	2	0	0
χ_{27}	0	−1	2	2	0	−4	2	−3	−4	2	1	−1
χ_{28}	0	0	−20	0	−1	0	0	2	2	2	2	0
χ_{29}	3	−2	0	0	−2	3	0	1	0	0	0	0
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	2	1	5	0	0	−4	−1	0	−10	−1	0	−1
χ_{32}	0	0	5	0	0	2	2	−3	10	1	0	0
χ_{33}	0	0	5	0	0	−1	−4	−3	−5	−2	0	0
χ_{34}	0	2	−20	0	0	0	0	−4	−10	2	0	2
χ_{35}	−3	0	15	0	−1	3	0	0	3	0	−2	0
χ_{36}	−1	0	0	0	1	1	1	−4	0	0	0	0
χ_{37}	−3	0	12	2	−2	−3	0	3	−3	0	2	0
χ_{38}	4	−1	0	0	2	−2	1	3	0	0	0	−1
χ_{39}	0	0	−26	−1	0	4	−2	0	11	−1	1	0

Table 2 The integer-valued character table of Lyons group Ly where $B_n = na \cup nb$ for $n = 21, 22, 33, 37, 40, 42$, $C_{24} = 24b \cup 24c$, $D_{31} = 31a \cup 31b \cup 31c \cup 31d \cup 31e$ and $E_{67} = 67a \cup 67b \cup 67c$ are unmatured dominat class

$C_{Ly}^{\mathbb{Q}}$	$20a$	$B21$	$B22$	$24a$	$C24$	$25a$	$28a$	$30a$	$30b$	$D31$	$B33$	$B37$	$B40$	$B42$	$E67$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	−2	1	0	0	0	0	−4	2	0	−1	2	0	2	2
χ_3	1	1	0	1	0	−1	−2	−1	2	0	0	−1	−1	1	0
χ_4	2	0	−1	−2	0	−2	0	2	2	0	−1	0	2	0	2
χ_5	0	0	0	0	0	−2	0	0	0	2	1	−2	0	0	0
χ_6	−1	0	0	0	−1	1	0	−1	−4	1	0	0	−1	0	0
χ_7	−1	0	1	−1	−1	0	−1	3	3	0	1	0	−1	0	0
χ_8	1	−1	0	0	−1	0	2	−1	2	0	0	−1	1	−1	−1
χ_9	1	0	−1	1	0	1	−1	3	0	0	−1	1	−1	0	0
χ_{10}	−2	1	−1	0	0	0	1	0	0	0	−1	0	0	1	0
χ_{11}	0	1	0		1	0	0	1	−2	0	0	0	0	−1	0
χ_{12}	0	0	1	0	0	−1	0	0	−3	−1	1	0	0	0	0
χ_{13}	−1	0	−1	0	1	0	0	−1	−1	0	1	0	1	0	0
χ_{14}	−2	0	0	0	0	0	0	−2	1	0	−1	0	0	0	0
χ_{15}	−1	0	0	−1	0	1	0	−1	2	0	0	0	1	0	0
χ_{16}	1	−1	0	0	1	0	2	0	0	0	0	0	−1	−1	0
χ_{17}	1	−1	0	2	0	1	−1	0	0	0	0	0	1	−1	−1
χ_{18}	2	1	0	−2	0	2	−2	0	0	0	0	0	2	1	−2
χ_{19}	2	1	0	0	−1	0	1	−1	−1	0	−1	0	0	1	0
χ_{20}	0	−2	0	0	0	0	−2	0	0	0	0	1	0	−2	2
χ_{21}	0	0	0	0	0	−3	0	0	0	0	0	0	0	0	1
χ_{22}	0	−1	0	0	0	0	0	2	−4	0	0	0	0	1	2
χ_{23}	0	2	0	0	0	0	0	−4	2	0	0	0	0	−2	2
χ_{24}	1	0	0	0	1	0	0	1	1	0	0	−1	−1	0	0
χ_{25}	1	0	0	−1	0	0	0	−1	2	0	0	0	−1	0	0
χ_{26}	0	−1	−1	0	0	0	0	0	0	0	1	0	0	1	0
χ_{27}	−2	0	0	0	0	0	1	2	2	0	0	0	0	0	0
χ_{28}	0	1	1	0	0	0	0	−2	−2	0	−1	0	0	−1	0
χ_{29}	0	1	0	1	0	0	1	0	0	−1	1	0	0	1	0
χ_{30}	0	0	0	0	0	−5	0	0	0	1	0	5	0	0	−5
χ_{31}	1	0	0	0	−1	0	0	2	−1	0	0	0	1	0	1
χ_{32}	1	0	0	−2	0	1	1	2	−1	0	0	0	−1	0	0
χ_{33}	1	0	0	1	0	1	1	−1	2	0	0	0	−1	0	0
χ_{34}	0	−2	0	0	0	0	0	−2	−2	0	0	−2	0	2	0
χ_{35}	−1	0	1	1	0	0	0	3	0	0	−1	−1	1	0	0
χ_{36}	0	1	−1	1	−1	0	0	0	0	0	1	0	0	−1	1
χ_{37}	0	0	0	−1	0	1	−1	−3	0	0	1	0	0	0	0
χ_{38}	0	0	0	0	1	0	−1	0	0	0	−1	−1	0	0	0
χ_{39}	−2	0	0	0	0	1	0	1	1	0	0	0	0	0	0

Acknowledgments The authors would like to thank the referees and editors for his/her useful comments and suggestions.

References

1. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, (Oxford Univ Press (Clarendon), Oxford, 1985)
2. W. Feit, G.M. Seitz, On finite rational groups and related topics. Ill. J. Math. **33**, 103–131 (1988)
3. S. Fujita, Enumeration of non-rigid molecules by means of unit subduced cycle indices. Theor. Chim. Acta. **77**, 307–321 (1990)
4. S. Fujita, Dominant representations and a markaracter table for a group of finite order. Theor. Chim. Acta. **91**, 291–314 (1995)
5. S. Fujita, Maturity of finite groups. An application to combinatorial enumeration of isomers. Bull. Chem. Soc. Jpn. **71**, 2071–2080 (1998)
6. S. Fujita, Inherent automorphism and $\mathbb{Q}$-conjugacy character tables of finite groups. An application to combinatorial enumeration of isomers. Bull. Chem. Soc. Jpn. **71**, 2309–2321 (1998)
7. S. Fujita, Direct subduction of $\mathbb{Q}$-conjugacy representations to give characteristic monomials for combinatorial enumeration. Theor. Chem. Acc. **99**, 404–410 (1998)
8. S. Fujita, Mark tables and $\mathbb{Q}$-conjugacy character tables for cyclic groups. An application to combinatorial enumeration. Bull. Chem. Soc. Jpn. **71**, 1587–1596 (1998)
9. S. Fujita, A simple method for enumeration of non-rigid isomers. An application of characteristic monomials. Bull. Chem. Soc. Jpn. **72**, 2403–2407 (1999)
10. S. Fujita, The unit-subduced-cycle-index methods and the characteristic-monomial method. Their relationship as group-theoretical tools for chemical combinatorics. J. Math. Chem. **30**, 249–270 (2001)
11. S. Fujita, Diagrammatical approach to molecular symmetry and enumeration of stereoisomers. Book Rev. N. Trinajstic. Croat. Chem. Acta **81**(2), A27–A28 (2008)
12. GAP Groups, *Algorithms and Programming, Lehrstuhl De fr Mathematik* (RWTH, Aachen, 1995)
13. A. Kerber, K. Thurlings, *Combinatorial Theory* (Springer, Berlin, 1982)
14. A. Kerber, *Applied Finite Group Actions* (Springer, Berlin, 1999)
15. D. Kletzing, Structure and representations of $\mathbb{Q}$-group. Lecture Notes in Math. 1084, Springer, 1984
16. H. Sharifi, Rational groups and integer-valued characters of Thompson group Th. J. Math. Chem. **49**, 1416–1423 (2011)